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The Black-Scholes Model Can Overvalue ESOs and Other Options

By Martin Greene, CPA/ABV, ASA

The Black-Scholes model[1] (BSM) is one of the most common formulas used in business valuation today. It is often used to value employee stock options and junior rounds of equity. Moreover, accounting guidance issued by the Financial Accounting Standards Board (FASB) indicates that the BSM is a valuation technique that meets the criteria required under Accounting Standards Codification (ASC) 718 for estimating the fair values of employee share options and similar instruments.[2]

However, the BSM was not originally intended for many present-day uses. Frequently, courts, accountants, attorneys, and other parties accept reports using the BSM without understanding the model's shortcomings. Without considering the underlying assumptions of the model, results can *overstate* options with longer exercise terms. Hence, using this formula to value public or privately held equity issued as compensation will likely overstate the compensation expense.

Normality issue. One challenge business valuators face is that the BSM was not designed for long-term expirations, and what applies in the public markets may not always be interchangeable with privately held companies. Furthermore, the BSM makes a number of assumptions that will not always be true regarding valuing privately held stock and nonmarketable stock options.[3]

One assumption is that it calculates the option price assuming stock returns are expressed as geometric rates of return that are *normally distributed*. A normal distribution[4] yields a bell-shaped curve, which many assume to be perfectly symmetrical, with each half mirroring the other, although that may not actually be the case. Although the BSM assumes a normal distribution, it does not test whether the variables produce a normal distribution. Hence, the conclusion may not be useful to measure the option value.

Monte Carlo analysis. In this article, to illustrate the BSM distribution, a Monte Carlo simulation (MCS) model of a dataset containing 100,000 trials was applied using Excel.[5] MCS may be considered ominous to some and conceptually apart from the BSM. However, the two can be very analogous. Both the BSM and the MCS can produce the same or similar option values over the relative

1 The Black-Scholes model was first published in 1973 by Fischer Black, a mathematician, and Myron S. Scholes, a professor of finance (they later sought the assistance of Robert C. Merton of MIT). See Fischer Black and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, Vol. 81, No. 3 (May-June, 1973), pp. 637-654.

2 ASC 718-10-55-16.

3 In addition to these, several other assumptions are used in the BSM.

4 A normal distribution measures the probability of an event happening and assumes most outcomes nearer the center and symmetrical, creating the bell shape. In a probability distribution, values must be between 0 and 1, relating to the likelihood of the event. Frequently, the market prices of publicly traded stock returns can reasonably follow this distribution.

5 While several MCS software applications are commercially available, Excel is readily available and allows the user to analyze individual trials.

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range[6] applying the same variables. However, the MCS can provide greater transparency in visually observing individual trials, which enables both the analyst and user to better understand any distortions in the calculated future stock price and option value. Much of this analysis focuses on the future stock price and not solely the option value.

The MCS shows that, for longer time periods until exercise, the standard deviation increases significantly and much of the option value is obtained from statistical outliers. In addition, simply to achieve the option price suggested by the BSM would require the stock price at the exercise date to be higher than would be reasonably expected for most every company. Statistical outliers can be outcomes with Z-scores[7] greater than three standard deviations from the mean.

This article is intended to provide assistance to valuation experts and others who use option values calculated with BSM to analyze the distribution, demonstrating whether the BSM variables produce a normal distribution. Experts should consider the following steps, as well as others: (1) construct an MCS based on the same variables as the BSM; (2) observe the outcomes on a histogram; (3) test the dataset for symmetry distribution called skewness; and (4) determine whether the dataset is peaked or flat relative to a normal distribution, which is known as kurtosis. These steps will be explained in this article.

If the dataset is not normally distributed, valuation experts may consider adjusting the dataset, such as determining a range of future stock prices and considering whether the adjusted data are

6 However, it should be noted, over much longer time periods, these models' option prices do vary. Eventually, the BSM option value approaches the current stock price, and the MCS option price can vary significantly.

7 A Z score is computed by subtracting the mean of a dataset from a raw data score or single outcome then dividing it by the dataset's standard deviation, which yields the number of standard deviations an outcome is away from the mean. The resulting Z score is then multiplied by the normal distribution function to determine its probability.

more normally distributed. It will be demonstrated that modifying the future stock price generally can improve the distribution but decreases the option value. This can be especially appropriate

Sidebar 1. Black-Scholes Model

The Black-Scholes formula calculates the price of a European option[1] using five inputs: (1) value of the underlying asset; (2) exercise price; (3) time to expiration; (4) implied volatility of the underlying asset; and (5) the risk-free interest rate:

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

Where,

$$d_1 = \frac{Ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{Ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

C = the theoretical call premium
N(x) = the cumulative distribution function of the standard normal distribution
T = the time to maturity
S = the spot or current price of the underlying asset
K = the exercise price
r = the risk-free rate (annual rate, expressed in terms of CCRR)
σ = the volatility of lognormal returns of the underlying asset. Commonly, it is believed that volatility is related to the stock price; however, typically it relates to the expected return, or r.
Ln = the natural logarithm
e = the exponential term (2.7183)

The following is a brief explanation and calculation of a BSM. It is presented to help the reader understand the model and see that it does not test the distribution's shape. As discussed throughout the text, option values are based on both time and volatility. To simplify the explanation, time below is being ignored, *or assumed to be one year*. If the model is to be applied to another period or if volatility is expressed other than annually, time must be considered in the computation:

The expression $SN(d_1)$ is the rate of growth for the stock price to be at or exceed the exercise price when the option expires. It is the expected value of the stock, multiplied by the probability that the stock price will be at or above the exercise price. $N(d_1)$ is future value of the stock. *If and only if*, the stock is above the exercise price at expiration, it is a conditional probability. The equation for d_1 is a Z-score, multiplied by N, the normal distribution function. In the expression d_1, Ln (K/S) is the rate of growth required for the stock price to reach the exercise price at expiration, which is the Z-score outcome. The symbol r is the Z-score mean. Generally, in finance r is reduced by the drag placed on it by the volatility or standard deviation. The standard deviation (volatility) is converted to the variance (by multiplying it to the second power) and then dividing by 2, which provides an estimate of a geometric return. However, in the expression $N(d_1)$, the volatility is added rather than subtracted.

The formula $N(d_2)$ is the probability that the stock price will be at or above the strike price when the option expires, meaning the option will be exercised. It is calculated similarly to $N(d_1)$ except symbol r is reduced for the drag placed on the return by the volatility. There is a relationship between d_1 and d_2. In the example below, if the volatility is added to d_2, it will agree to d_1. Remember, the BSM adjusts for time but has been ignored (or assumed to be 1) in this very brief explanation.

An example of applying the BSM formula to calculate an option value is as follows: Assume: (1) a current stock price of $30.00 per share; (2) an exercise price of $35.00 per share; (3) no dividends; and (4) an r of 2% (adjusted for CCRR). These variables have been assumed throughout this article, but volatility and time will vary. Furthermore, volatility will be expressed on an annualized basis. Below it is assumed to be 80%:

Description		Formula		Computation		Value
Outcome	=	LN(S/K)	=	LN($30.00$35.00)	=	(0.1542)
Drift (mean)[1]	=	r+σ^2/2	=	1.98 + 80%^2/2	=	0.3398
d1	=	LN(S/K) + (r+σ^2/2)/σ	=	(-0.1542+.3397)/80%	=	0.2321
Drift (mean)	=	r-σ^2/2	=	1.98 - 80%^2/2	=	(0.3002)
d2	=	LN(S/K) + (r-σ^2/2)/σ	=	(-0.1542+ -.3002)/80%	=	-0.5679
Nd1	=	0.2319			=	0.5918
Nd2	=	-0.5681			=	0.2850
C		$SN(d_1) – Ke^{-rt}N(d_2)$		30*.5918-35*2.7183^-0.0198*.2850	=	**7.97**

1 adjusts drift for the standard deviation

1 A European option is exercisable on the expiration date (at the end of its term).

Reprinted with permissions from Business Valuation Resources, LLC for privately held businesses as they usually will operate only within certain ranges because numerous factors limit a privately held company's ability for growth and may keep it from achieving the future stock prices assumed by the BSM. Furthermore, business valuators may have determined the company's value at the exercise date by using other valuation approaches rather than reliance on statistical outliers or random numbers. However, each analyst should also consider the appropriate method to test the model or alternative methods.

The MCS uses the same variables as the BSM: current stock, price exercise, holding period, a risk-free rate, and volatility. An example of applying the BSM is presented in Sidebar 1. The variables assumed are: (1) a current stock price

Sidebar 2. Calculation of Future Stock Price Used in the MCS

The formula used to model the stock price behavior in the BSM is rewritten and is also presented in a table below. However, it calculates the future stock price using random variables. In this table, it demonstrates how random variables can influence increase stock prices in the BSM. This is the current stock price times the exponent of a log return, which is the current price times the exponent (mean drift plus the V_T times the random variable). It can be expressed in two components: (1) a deterministic component; and (2) a stochastic component. The deterministic portion is the drift, which is calculated by the natural log of the CCRR multiplied by time. The formula for drift is equal to LN(CCRR + 1) multiplied by the number of years until the exercise date. Volatility is scaled to time (). Mean drift is the drift adjusted by one-half of the variance of $V_{T.}$

To calculate the stochastic component to determine the future stock prices, the exponent of the probability distribution is multiplied by the volatility plus the drift times the current stock price.

These computations are presented to show that, with low random variables and longer holding periods, the future stock price at exercise increases quickly, resulting in values that may not really be obtainable. In the exhibit, 60% volatility was used for the five years. The table reflects the distributions range, from -3 to +3, to compare outcomes.

Under the assumption of a normal distribution, the median value of future stock prices is $13.45. However, under the BSM, the future stock price would be calculated by stock price times natural logarithm of the r and time, or $30 * EXP$^{(2\% * 5\ years)}$ = $33.15. If the mean is greater than the median, this indicates that the curve is positively skewed.

Further, the future stock price (assuming a probability distribution of 3) grew from $30 per share to over $750 in five years, which represents a CAGR of 90%. Is this likely a reasonable assumption? It also is helpful to consider the value a probability distribution of -3, which reflects a stock price decline to $0.24 in five years. Furthermore, it should be noted that it is close to zero but still positive.

If the assumptions are changed (such as shortening the holding periods), median increases and the mean and median are much closer and the curve is more bell-shaped, with less skewness and greater reliability. In a normal distribution with a symmetrical, bell-shaped curve, the mean, median, and mode all agree.

Calculation of the future stock price

Assumptions:			
Initial stock	S	$	30.00
Year	T		5.0
Risk-free rate	RFR		2.00%
Risk-free rate continuous compounded	CCRR		1.98%
Volatility (annual)	σ		60%

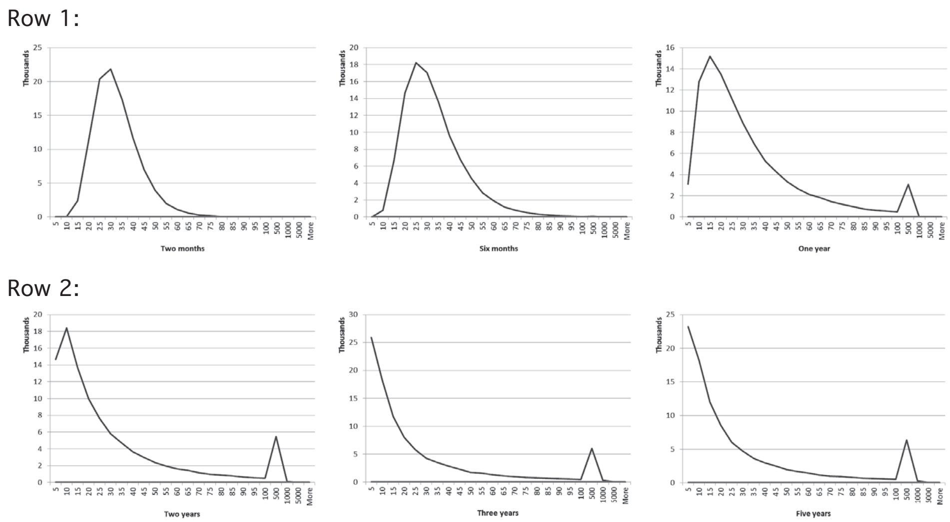
Drift	Volatility V_T	Drift (mean)	Random variable	Future stock price
ln(CCRR+1)*T	σ*sqrt(T)	Drift-.5V_T^2		[1]
9.8046%	134.16%	-80.1954%	3	753.0753
9.8046%	134.16%	-80.1954%	2	196.8662
9.8046%	134.16%	-80.1954%	1	51.4641
9.8046%	134.16%	-80.1954%	0	13.4535
9.8046%	134.16%	-80.1954%	-1	3.5170
9.8046%	134.16%	-80.1954%	-2	0.9194
9.8046%	134.16%	-80.1954%	-3	0.2403

[1] The natural logarithm (2.71828) to the power of volatility (V_T) times the random variable plus Drift (mean), then all is multiplied by the initial stock price (S).

of $30.00 per share; (2) an exercise price of $35.00 per share; (3) no dividends; and (4) a risk-free rate of 2% (adjusted for continuous compounded rate of return (CCRR), shown as (r)). Furthermore, these variables will be assumed throughout this article not only in the BSM calculation, but also in the MCS, but volatility and time will vary. Also, volatility will be expressed on an annualized basis. The Sidebar 1 analysis is also helpful in pointing out that the BSM does not test the normality of the distribution.

Exhibit 1. Histograms of Outcomes From Two Months to Five Years

Assumptions:			
Initial stock	$30.00	r adjusted for CCRR)	2.00%
Exercise price	$35.00	Volatility (annual)	80%
Years	.167-5.000		



Using the MCS to calculate value. Rather than calculating a Z-score for an entire distribution, as in the BSM, the MCS calculates the stock price at exercise date and option value for each trial. According to the Central Limit Theorem, as the number of trials approaches infinity, the outcomes would be expected to approximate a bell-shaped curve, as assumed with the BSM, based on certain defined assumptions. The option price under the MSC reasonably agrees with that of the BSM. A bell-shaped curve assumes outcomes are generally distributed between -3 and +3 standard deviations (after +3) with most of the variables closest to zero. The empirical rule tells us that 99.7% of the outcomes should fall within these six standard deviations of the mean (three positive and three negative). Consequently, this indicates that fewer than 200 outcomes should be greater than three standard deviations on each of a two-tailed distribution of 100,000 trials.

The MCS calculates the stock price at exercise date by applying a random variable to each trial. Then the exercise price is subtracted, the present value is calculated for each option value in the money, and then the outcomes are averaged. As the time to the exercise date and/or volatility increases, the stock price at exercise date can increase significantly. To clarify, stock prices at the exercise date are calculated applying several random variables (see Sidebar 2).

Histogram of outputs. Presented here is the MCS of the dataset of stock prices at the exercise run for several durations ranging from two months to five years using an 80% volatility. The outcomes are plotted in Exhibit 1.

In Row 1, the outcomes range from two months up to one year, and the distributions appear relatively bell-shaped but skewed to the right, or positively skewed, which is a characteristic of a lognormal distribution. Skewness is the measure of symmetry of the probability distribution around the mean, which can be positive or negative. Lognormal distributions will be positively skewed, having a longer tail on the right side and greater mass on the left side. The reason is that higher positive random variables produce larger stock prices at the exercise date, but larger negative random variables do not produce negative values. As negative log return approaches negative infinity, the value when converted to an integer must be greater than zero. This is useful in valuing stock because the price cannot be lower than zero.

A second observable characteristic in the curves from the two-month-to-one-year range is kurtosis, which measures the peakedness of the curve. Although it is not completely clear from looking at the histograms, in the two-month curve, as the stock prices seem to reach the x-axis at approximately $75 (only approximately 100 outcomes of the 100,000 are greater than $75) and, in the six-month curve, the stock price reaches the x-axis at about $100 (again, only approximately 100 outcomes are greater than $100), it increases to a stock price of almost $1,000 by one year for approximately 3,000 of the trials. The bump in the curve from outcomes is due to the limitation of the graphing as all values for outcomes between $100 and $1,000 are placed in the same histogram category.

These curves reflect higher kurtosis, which has a higher, narrower peak and heavier, fatter tails, and their outcomes are more spread out; they have more of a tendency to populate the extremes, resulting in a less reliable conclusion. Kurtosis risk applies when it is assumed the distribution is bell-shaped, but many observations do not cluster near the mean (often referred to as "fat tails"). Ignoring kurtosis risk will cause any model to understate the risk of variables with high kurtosis.

In positively skewed outcomes, one end is wider than in a normal distribution. These are called leptokurtosis (also known as "fat tailed"). A fat tailed distribution can result in an excess number of outcomes or higher future stock prices at the exercise date and a higher option price. Hence, the shape of the curve changes and is less bell-shaped. Moreover, more and large outcomes are further from the mean, increasing the stock price at the exercise date. Although the BSM outcomes remain more bell-shaped when applied to shorter holding periods, the BSM outcomes have been criticized for what is called kurtosis risk.

In Row 2, the histograms are from two, three, and five years until exercise and all three histograms appear as slopes. They all seem to reach the x-axis at stock prices of $500 to $1,000. In the simulation above, the number of outcomes from $500 to $1,000 was approximately 5,500, 6,000, and 6,500, respectively. They each had several hundred stock prices over $1,000 and approximately 50 over $5,000. It will be clearer looking at the summary of simulations presented shortly that much of the option value is coming from these outliers.

As business valuators and other users of the BSM who are familiar with clients, is it reasonable for outcomes of a $30 share price to increase to $500 in five years, even if the random numbers suggest it?

Observation of MCS outcomes. It would be helpful but not feasible to present the entire simulations for the respective time periods to observe changes in several important attributes modeled. However, the chart in Exhibit 2 presents a summary of important attributes from a single simulation for two volatility respective time periods.[8]

In Line 1 of Exhibit 2, both option prices under the BSM and MSC are fairly close, which supports using the MCS to test the BSM. Since the MCS recalculates a value with every simulation, the values will change with each simulation. The variations are larger for longer holding periods and higher volatilities. For example, in five years, the change in option price can be up to $0.50 per simulation with the above assumptions when compared with the BSM but will be only a few cents for the shorter periods. As the number of trials increase, the difference can narrowed.

Stock price and standard deviation. In Lines 2 and 3, the exhibit presents the average stock price at the exercise date calculated applying the formula stock price times natural logarithm (r * the time until exercise) and then comparing to the average stock price at the exercise date from the simulations. The outcomes are fairly close,

8 Each simulation makes a small change in the outcome. The chart in this article presents only one single simulation for period and volatility.

Exhibit 2. Summary of Monte Carlo Simulation

	Summary of Monte Carlo Simulation						Exhibit II
	Number of years until exercise	5	4	3	2	1	0.5
1	Option value with MCS	$ 18.56	$ 16.81	$ 14.65	$ 11.96	$ 7.95	$ 5.11
	Black Scholes option value	$ 18.36	$ 16.55	$ 14.76	$ 11.87	$ 7.97	$ 5.10
2	Average stock price at exercise date - calculated	$ 33.16	$ 32.50	$ 31.86	$ 31.22	$ 30.61	$ 30.30
3	Average stock price at exercise date MSC	$ 32.90	$ 32.14	$ 31.99	$ 31.37	$ 30.54	$ 30.34
4	Volatility	80%	80%	80%	80%	80%	80%
	Standard deviation	147.82	105.00	76.39	50.44	29.26	18.57
5	Volatility	40%	40%	40%	40%	40%	40%
	Standard deviation	36.10	30.93	25.10	19.23	12.71	8.73
	Volatility	80%	80%	80%	80%	80%	80%
6	Number of values over the exercise price	17,530	19,591	22,397	25,403	28,261	29,538
7	Number of value greater than three standard deviations	909	1,128	1,340	1,640	1,774	1,708
8	Average stock price of outcomes greater than three standard deviations	$ 1,020	$ 701	$ 469	$ 300	$ 165	$ 105
9	Skewness	34.5	28.7	17.4	7.7	3.6	2.1

and increasing the number of trials per simulation could improve the result. Volatility does not affect the stock price as seen by the formula.

In Lines 4 and 5, the exhibit presents the standard deviation for each simulation at the various time periods and for 80% and 40% volatilities. It shows that larger volatilities and a longer time until exercise significantly increase standard deviation, and, since the negative outcomes cannot be lower than zero, more values are further from the mean. As we saw previously, the mean of the stock price at exercise, regardless of the volatility, increases only by r, but, as the volatilities and time increase, the standard deviation in the table increases significantly and values are farther away from the mean. Many more outcomes are closer to zero, and many positive outcomes can be significantly away from the mean, forcing the curve to be less bell-shaped.

For both the 80% and 40% volatilities, the standard deviation increased from 18.57 to 147.82 and from 8.73 to 36.10, respectively. Even using a 40% volatility, the value of three standard deviations is a compound annual growth rate (CAGR) of over 33% in five years.

While the number of values over the exercise price decreases for an 80% volatility (Line 6), the number of outcomes greater than three standard deviations away from the mean after five years is approximately 0.09% (900/100,000) (Line 7). Statistically, it should be under 200, or below 0.02%, and the average value of the outcomes is over $1,000 (Line 7), or six standard deviations from the mean. Much of the stock price or option value is obtained from these 900 outcomes with stock prices over $1,000 after a five-year holding period. Would it be expected for a stock in the above example to have more than a 100% CAGR in five years and five times the number of outcomes greater than three standard deviations than a normal distribution?

Some pundits suggest excluding outcomes greater than three standard deviations. Using MCS, selected outcomes can be eliminated or adjusted. Applying MCS to the first example above, and excluding outcomes greater than the three standard deviations (assuming an exercise date five years in the future and an 80% volatility), the option price decreases from approximately $18.50 to approximately $10.00. Most business valuators can provide some insight as to the appropriate future stock price for their subject company rather than solely relying on the formula.

For a holding period of six months, approximately 1,700 outcomes are greater than three standard deviations away from the mean (but the expected number of outcomes should still be under 200). However, during a short time period, the standard deviation of the percentage change in price is usually much greater than its expected percentage change.

Skewness. A formula to measure skewness was applied to the data in Line 9. Using a holding period of six months and volatility of 80%, the skewness was just over 2.00 and rose to 34.5 after five years. As a rule of thumb, skewness should be no greater than 2.0 to be more normally distributed; therefore, for the shorter periods, the model appears to be normally distributed. There was much variation in each simulation in the skewness for the longer holding period. While a calculation can be applied to kurtosis, the scores for the longer holding periods were not meaningful.

In the above computation, an 80% volatility was applied, which was used for demonstration purposes. Some may think that, if the volatility were lowered, the BSM would avoid the issues presented in the article. Even assuming a 40% volatility and using a holding period until exercise of five years, the option price value is approximately $10.00. The skewness is approximately 4.5. Limiting the outcomes to three standard deviations, the option price is approximately $7.00, a decrease of 30%, and the skewness drops below 2. The standard deviation declines from approximately $37.00 to around $26.00.

Conclusion. A limitation of the BSM is that it does not test distribution properties. This can be very important. Without testing this underlying assumption that the BSM outcome is normally distributed, can one be certain the BSM is providing a reasonable conclusion? Like many formulas, its value is limited to the quality of the input and the skill of the analyst. It should be noted that, when applying the BSM to shorter time periods for the exercise date, even for higher volatilities, the distributions remain sufficiently normal. For longer holding periods, this will impact the model's shape and integrity. Therefore, it appears that the BSM is not appropriate for longer holding periods without modification. Business valuators, accountants, and other analysts may want to consider applying the BSM only after an analysis of the underlying distribution assumptions and their conclusions. Frequently, an option value will be used by a business valuator who estimates a future value under the income approach. Considering whether a subject company can achieve each future stock price appears to be more consistent and supportive than the formative approach without any analysis. ◆

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